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Conical flow near singular rays

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The steady flow of an ideal gas past a conical body is investigated by the method of matched asymptotic expansions, with particular emphasis on the flow near the singular ray occurring in linearized theory. The first-order problem governing the flow in this region is formulated, leading to the equation of Kuo, and an approximate solution is obtained in the case of compressive flow behind the main front. This solution is compared with the results of previous investigations with a view to assessing the applicability of the Lighthill–Whitham theories.

1. Introduction and mathematical formulation

1.1. *Introduction*

The problem to be studied here is the supersonic flow of an ideal gas about a stationary body of arbitrary conical shape. The investigation, motivated by recent interest in sonic-boom propagation, considers what is perhaps the simplest example of a situation in which essentially multi-dimensional effects near the wave fronts of linearized theory preclude the application of the standard co-ordinate straining and shock fitting techniques, a fact pointed out by Hayes *et al.* (1971). The geometry of the problem is shown in figure 1.

Even with the restriction to conical bodies, exact solutions to the nonlinear equations of motion are known for only a few simple body shapes; for more general shapes one must resort to approximate analytical or numerical methods (Courant & Friedrichs 1967). Analytically, progress can be made by assuming that the body produces only small disturbances in an otherwise uniform stream, which leads directly to the linearized theory of supersonic flow developed in Ward (1955), where a number of specific problems are solved.

However, the results predicted by linear theory are not uniformly valid, as Lighthill (1949*b*) and Whitham (1952, 1956) point out. In particular, they predict a continuous field where surfaces of discontinuity (shocks) are known to exist. The limitations of the theory appear more clearly when one regards a linear solution as the first term of an asymptotic expansion of a solution to the full nonlinear field equations governing the flow. In fact, this expansion, which will subsequently be referred to as the 'outer expansion', does not converge uniformly in any region containing the linearized wave fronts (Lighthill 1949*b*).

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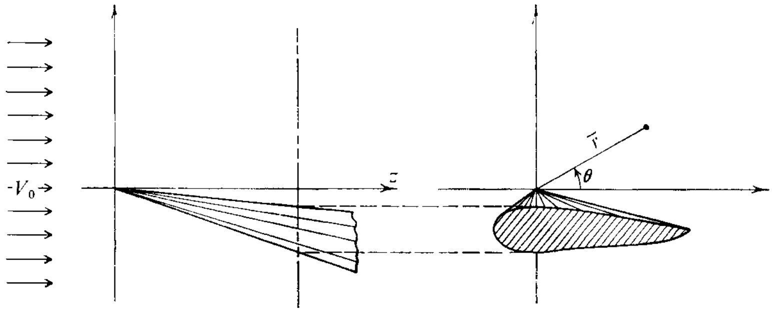

FIGURE 1. Geometry of typical conical body, and co-ordinate system.

To overcome these difficulties, Lighthill (1949*a*, *b*) and Whitham (1952, 1956) devised special asymptotic methods, and applied them to a number of problems. In particular, Lighthill (1949*b*) discussed flow past conical bodies extensively. The great advantage of the special techniques is that they yield the first term of a uniform or composite asymptotic expansion, valid near the linearized wave fronts as well as away from them, while requiring only a solution of the equations of the linearized theory. They include the prediction of shocks in regions where they are expected.

However, though the Lighthill–Whitham results seem to give a correct first-order description of the flow over the major portion of the wave fronts, they are obviously incorrect along singular rays (i.e. lines along which the linearized plane characteristics of discontinuity are tangential to the characteristic cone emanating from the apex). The shock strength, as predicted by the Lighthill–Whitham theories, approaches infinity near such a line. One reason for this non-uniformity is that both Lighthill and Whitham used approximate representations of the solutions to the linearized problem that are not valid near the singular rays, a problem which may be overcome by using asymptotic representations of the solutions to the linearized problem that are uniform in a parameter measuring position along the wave front. Such representations may be obtained either by expanding known solutions, or by the methods outlined in Myers (1971). However, even with such representations available, it is not clear how the methods of Lighthill and Whitham could be extended to yield first-order solutions to the nonlinear problem valid near the singular ray. Some attempts have been made. Legras (1953) tried to generalize Lighthill's approach, while Davis (1969) attempted to extend the Whitham technique. Their conclusions will be examined in the light of results obtained in this paper.

The mathematical technique of matched asymptotic expansions, which is explained by Van Dyke (1964), will be applied to study the flow near the singular rays. This method permits a systematic, precise formulation of the equations and auxiliary conditions governing the flow near the singular ray, which may be used to deduce the solution or to check solutions obtained by other methods. It leads, however, in contrast to the Lighthill–Whitham methods, to a nonlinear first-order problem. In fact, the equations and shock conditions derived are equivalent to those found less formally by Kuo (1955).

1.2. *Mathematical formulation*

Consider steady flow of an ideal, non-conducting gas past a body. It is assumed that the gas is polytropic, with constant specific heats. The motion is governed by the equations of inviscid gas dynamics derived, for example, in Courant & Friedrichs (1967):

$$\left.\begin{aligned} &\overline{\nabla}.(\bar{\rho}\bar{\mathbf{v}}) = 0,\\ &\bar{\mathbf{v}}.\nabla(\tfrac{1}{2}\bar{\mathbf{v}}.\bar{\mathbf{v}}+\gamma\bar{p}/(\gamma-1)\bar{\rho}) = 0,\\ &\bar{\mathbf{v}}.\nabla\bar{\mathbf{v}}+(\nabla\bar{p})/\bar{\rho} = 0, \end{aligned}\right\} \qquad (1.1)$$

where $\bar{\mathbf{v}}$, $\bar{p}$, and $\bar{\rho}$ are velocity, pressure and density, respectively.

Let $(\bar{r},\theta,z)$ be a cylindrical co-ordinate system and assume the flow ahead of the body is a uniform supersonic stream with $\bar{\mathbf{v}} = V_0\mathbf{e}_z$, $\bar{p} = p_0$, $\bar{\rho} = \rho_0$, where $\mathbf{e}_z$ is a unit vector in the z direction (figure 1). If the free-stream sound speed is denoted by $c_0 = (\gamma p_0/\rho_0)^{\frac{1}{2}} < V_0$, then we define $M = V_0/c_0$, $\beta = (M^2-1)^{\frac{1}{2}}$, and $r = \beta\bar{r}/z$. The surface of conical bodies with apex at the origin is described by

$$\hat{y} = G^*(\hat{x}) \quad \text{for} \quad \hat{x} = r\cos\theta, \quad \hat{y} = r\sin\theta. \qquad (1.2)$$

We define dimensionless perturbation variables as follows:

$$u = \frac{\bar{v}_{\bar{r}}}{V_0}, \quad v = \frac{\bar{v}_\theta}{V_0}, \quad w = \frac{\bar{v}_z - V_0}{V_0}, \quad p = \frac{\bar{p}-p_0}{c_0^2\rho_0}, \quad \rho = \frac{\bar{\rho}-\rho_0}{\rho_0},$$

where $\bar{v}_r$, $\bar{v}_\theta$ an $\bar{v}_z$ are the cylindrical velocity components, and define the (column) vector q of perturbation variables by

$$q^T = [u, v, w, p, \rho].$$

Since only conical bodies are to be considered, we introduce the assumption of conical flow, under which we seek solutions of the equations of motion which depend on r and θ but not on z. In such solutions, all flow variables are constant on straight lines through the origin. The equations of motion (1.1), for conical flow, yield the following quasi-linear vector equation for q (Zahalak 1972):

$$aq_r + bq_\theta + d = 0, \qquad (1.3)$$

where the subscripts denote partial differentiation, and the matrices appearing in (1.3) are defined by

$$a = \begin{bmatrix} r(1+\rho) & 0 & -r^2\beta^{-1}(1+\rho) & 0 & J \\ M^2J & 0 & 0 & r/(1+\rho) & 0 \\ 0 & M^2J & 0 & 0 & 0 \\ 0 & 0 & M^2J & -r^2/\beta(1+\rho) & 0 \\ \beta uIJ & \beta vIJ & \beta(1+w)IJ & \gamma\beta(1+\rho)J & -\beta(1+\gamma p)J \end{bmatrix},$$

$$b = \begin{bmatrix} 0 & 1+\rho & 0 & 0 & v \\ M^2v & 0 & 0 & 0 & 0 \\ 0 & M^2v & 0 & 1/(1+\rho) & 0 \\ 0 & 0 & M^2v & 0 & 0 \\ \beta uvI & \beta v^2I & \beta(1+w)vI & \beta\gamma v(1+\rho) & -\beta v(1+\gamma p) \end{bmatrix},$$

$$d^T = [u(1+\rho),\ -M^2v^2,\ M^2uv,\ 0,\ 0],$$

where $\quad I = M^2(\gamma-1)(1+\rho)^2 \quad \text{and} \quad J = ru - r^2\beta^{-1}(1+w).$

The boundary condition expressing the requirement that the velocity at the surface of the body be tangential to the surface may be formulated (Zahalak 1972) as

$$(\hat{x}^2+\hat{y}^2)^{-\frac{1}{2}}\{\beta(dG^*/d\hat{x})(\hat{x}u+\hat{y}v)-\beta(\hat{y}u+\hat{x}v)\}+(1+w)[G^*-\hat{x}(dG^*/d\hat{x})]=0 \quad (1.4)$$

on $\hat{y} = G^*(\hat{x})$. The flow is assumed to be undisturbed ahead of the wave surface generated by the body, yielding the initial condition $q \equiv 0$ for r sufficiently large.

Finally, as the flow is supersonic, we seek 'weak solutions' (in the sense of Courant & Hilbert 1966) of (1.3), which may be discontinuous or have discontinuous derivatives across certain conical surfaces. Derivatives of q may be discontinuous across a characteristic surface $r = 1-h(\theta)$, while q is itself continuous across such a surface, if $h(\theta)$ satisfies the characteristic equation (Zahalak 1972)

$$M\left\{\frac{1+\rho}{1+\gamma p}\right\}^{\frac{1}{2}}\{-\beta u(1-h)-\beta v h'+(1+w)(1-h)^2\}=\{\beta^2(1-h)^2+\beta^2h'^2+(1-h)^4\}^{\frac{1}{2}}. \quad (1.5)$$

Further, q may be discontinuous across shocks, $r = 1-\eta(\theta)$. On such surfaces the set of shock conditions to be satisfied may be taken as (Keller 1954)

$$\left.\begin{array}{c}[(1+\rho)v_n]=0, \quad [p+M^2(1+\rho)v_n^2]=0,\\ [n_z u-n_{\bar{r}}w]=0, \quad [n_\theta u-n_{\bar{r}}v]=0,\\ [(\gamma-1)M^2v_n^2+2(1+\gamma p)/(1+\rho)]=0,\end{array}\right\} \quad (1.6)$$

on $r = 1-\eta(\theta)$, where $v_n = n_{\bar{r}}u+n_\theta v+n_z(1+w)$, and where $\mathbf{n}$, the normal to the shock, is given by

$$\mathbf{n} = n_{\bar{r}}\mathbf{e}_{\bar{r}}+n_\theta\mathbf{e}_\theta+n_z\mathbf{e}_z = \frac{-\beta(1-\eta)\,\mathbf{e}_{\bar{r}}-\beta\eta'\mathbf{e}_\theta+(1-\eta)^2\mathbf{e}_z}{\{\beta^2(1-\eta)^2+\beta^2\eta'^2+(1-\eta)^4\}^{\frac{1}{2}}}.$$

In addition to the above shock conditions, it can be shown (Courant & Friedrichs 1967) that the pressure must increase across a shock in the direction of the mass flux.

In §§2 and 3 we briefly indicate how the flow in regions away from the singular ray, which has been extensively treated by other methods, may be analysed through the formalism of matched asymptotic expansions. In §4 we apply this formalism to obtain a description of the flow near the singular ray.

2. The outer expansion

Solution of the problem stated in §1 by the method of matched asymptotic expansions begins by introducing, through the boundary conditions, a perturbation parameter ϵ, which measures the magnitude of the disturbance generated by the body. Then $q = q(r,\theta,\epsilon)$, where, by definition, $q(r,\theta;0) = 0$.

The first step of the solution is to assume an 'outer expansion' of the form

$$q(r,\theta;\epsilon) = \delta_1(\epsilon)\,q^{(1)}(r,\theta)+\delta_2(\epsilon)\,q^{(2)}(r,\theta)+\dots, \quad (2.1)$$

where the $\delta_i(\epsilon)$ are diagonal matrices with diagonal elements $\delta_{ij}(\epsilon)$ ($j = 1,2,\dots,5$) and

$$q^{(j)T} = [u^{(j)},v^{(j)},w^{(j)},p^{(j)},\rho^{(j)}].$$

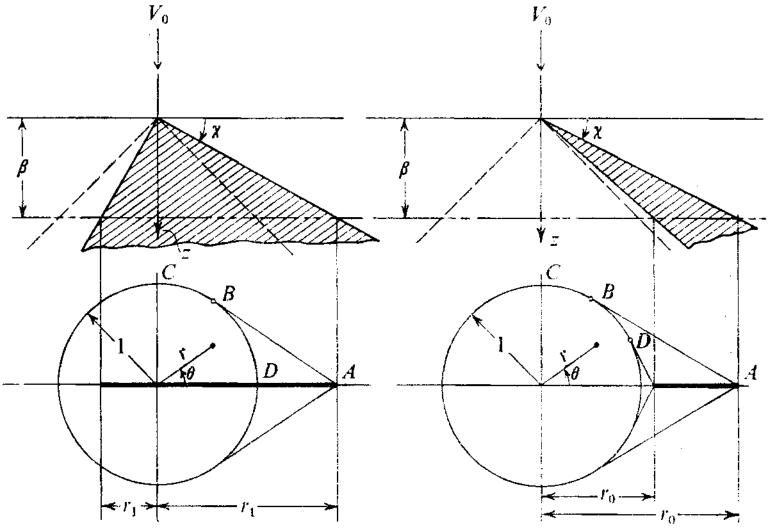

FIGURE 2. Typical characteristic geometry for first-order outer problem.

The sequences of gauge functions $\{\delta_{1j}(\epsilon), \delta_{2j}(\epsilon), \delta_{3j}(\epsilon), \ldots\}$ are assumed to be asymptotic sequences as defined in Erdélyi (1956) and Van Dyke (1964).

Gauge functions for the outer expansion are discussed in Ward (1955), where a distinction is drawn between 'thin bodies' and 'slender bodies'. Slender bodies do not generate the diffraction-like effects that are of special interest in this study, and can be treated satisfactorily by the methods of Lighthill (1949*a*, *b*) or Whitham (1952, 1956); they will not be considered further here. For thin conical bodies whose surface is given by $\bar{G}(\mathbf{x};\epsilon) = 0$, the mean body surface consists of planes, for simplicity assumed to be a portion of the plane $\theta = 0, \pi$. Therefore the body surface (see figure 2) is defined by

$$\hat{y} = \epsilon G(\hat{x}) \quad \text{for} \quad r_1 \leqslant \hat{x} \leqslant r_0. \tag{2.2}$$

To construct the first-order outer problem, only the first-order gauge functions are needed, and Ward (1955) indicates that for thin bodies these are all equal to ϵ itself. Therefore, for thin bodies,

$$\delta_{ij} = \{\epsilon, \delta_{2j}(\epsilon), \delta_{3j}(\epsilon), \ldots\} \quad (j = 1, 2, 3, 4, 5). \tag{2.3}$$

If (2.1) and (2.3) are inserted into the equations and auxiliary conditions stated in §1, and the outer limit taken, then, as shown in Zahalak (1972), there results a first-order problem which is completely equivalent to the linearized theory of supersonic conical flow. The solution of this problem may be reduced to the determination of a 'conical potential' $f(r, \theta)$, satisfying

$$(1-r^2)f_{rr} + \frac{1}{r}f_r + \frac{1}{r^2}f_{\theta\theta} = 0, \tag{2.4}$$

where $q^{(1)}$ is determined from f by

$$q^{(1)T} = \left[\beta\frac{\partial}{\partial r}, \ \frac{\beta}{r}\frac{\partial}{\partial \theta}, \ \left(1-r\frac{\partial}{\partial r}\right), \ -M^2\left(1-r\frac{\partial}{\partial r}\right), \ -M^2\left(1-r\frac{\partial}{\partial r}\right)\right]f.$$

The properties of the solutions to this first-order outer (linearized) problem are discussed at length in Lighthill (1949*b*), Ward (1955) and Zahalak (1972), and will be only briefly outlined here. Equation (2.4) is hyperbolic outside the unit circle $r = 1$ (corresponding to the Mach cone of the apex), and elliptic inside. In the hyperbolic region, the characteristics consist of straight lines tangential to the unit circle. Typical characteristic geometry for the first-order outer problem is shown in figure 2. $q^{(1)}$ vanishes in the region of undisturbed flow outside the wave front, which in the first-order outer representation consists of segments of the unit circle and straight lines through the leading edges tangential to the unit circle (e.g. $ABCA$ in figure 2).

Both f and its derivatives are continuous across the unit circle, which we call the 'diffracted front' by analogy with unsteady two-dimensional diffraction problems. However, the derivatives of f (and therefore the perturbation flow variables) may be discontinuous across characteristics through the leading edges of the body, which we call 'main fronts'. The main fronts are tangential to the unit circle at the singular points, e.g. point B in figure 2.

The outer expansion of the boundary condition (1.4) yields

$$v^{(1)}|_{\hat{y}=0} = v_0(\hat{x}) = \beta^{-1}\{G(\hat{x}) - \hat{x}G'(\hat{x})\} \quad \text{for} \quad r_1 < \hat{x} < r_0, \tag{2.5}$$

which shows that, according to linearized theory, there will be a discontinuity in $q^{(1)}$ across the main front provided that

$$G(r_0) - r_0 G'(r_0) \neq 0. \tag{2.6}$$

In fact, we have found it convenient to impose certain restrictions on the function G that describes the body shape and orientation, to limit somewhat the variety of possible flow patterns near the singular point. The first of these restrictions is (2.6). The second is

$$G''(r_0) \leqslant 0, \tag{2.7}$$

which excludes, for example, bodies with supersonic cusped leading edges. Further, we assume that $G(r)$ is analytic at $r = r_0$. These restrictions are satisfied by many bodies possessing shapes and orientations of practical interest.

Lighthill (1949*b*) and Zahalak (1972) showed that, if a characteristic connects a point on the unit circle to a point $r = r^*$ on $\theta = 0$, and if the boundary function (2.5), $v_0(r)$, is analytic at $r = r^*$, then $f(r, \theta)$ may be represented in the neighbourhood of this point by the two series

$$f(r,\theta) = D(\theta) - C(\theta)(1-r) + \begin{cases} \tfrac{2}{3}B(\theta)(r-1)^{\frac{3}{2}} + \dots & (r > 1) \\ -\tfrac{2}{3}A(\theta)(1-r)^{\frac{3}{2}} + \dots & (r < 1). \end{cases} \tag{2.8}$$

The notation here agrees with that of Lighthill (1949*b*).

However, if $r = 1, \theta = \theta_0$ is the point at which the characteristic through the leading edge is tangential to the unit circle; the representation (2.8) is no longer valid for $r < 1$, since the boundary conditions for the elliptic problem are discontinuous at the point. The behaviour of the first-order outer solution near such a point may be obtained by analytic function theory, or through the uniform wave-front expansions discussed in Myers (1971). The former method was employed Zahalak (1972) to deduce that, for $r < 1$,

$$w^{(1)}(s,\theta) = \{w^{(1)}(0,\theta_0^-)+g_1(s,\theta)\}\left\{\frac{1}{2}+\frac{1}{\pi}\tan^{-1}\left(\frac{\theta_0-\theta}{-s}\right)\right\}$$
$$+g_2(s,\theta)\log\{s^2+(\theta_0-\theta)^2\}^{\frac{1}{2}}+g_3(s,\theta), \qquad (2.9)$$

where $s = -\operatorname{sech}^{-1}(r)$. The $g_i(s,\theta)$ in (2.9) are analytic at $s = 0, \theta = \theta_0$ and vanish there. The velocity component $w^{(1)}$ satisfies Laplace's equation in the variables s and θ, and (2.9) reflects the logarithmically singular behaviour of such a solution near a point at which the boundary data are discontinuous. The remaining first-order components of q may be obtained from

$$\left.\begin{aligned}\frac{1}{\beta}u^{(1)} &= f_r = -\frac{1}{r}w^{(1)}(r,\theta)-\left\{\int_1^r\frac{w^{(1)}(r^*,\theta)}{r^{*2}}dr^*-D(\theta)\right\},\\ \frac{1}{\beta}v^{(1)} &= \frac{1}{r}f_\theta = -\left\{\int_1^r\frac{w_\theta^{(1)}(r^*,\theta)}{r^{*2}}dr^*-\frac{d}{d\theta}D(\theta)\right\},\\ p^{(1)} &= \rho^{(1)} = -M^2w^{(1)}.\end{aligned}\right\} \qquad (2.10)$$

The behaviour of the coefficients $A(\theta)$, $B(\theta)$, $C(\theta)$ and $D(\theta)$ in (2.8) near the singular point was also examined in Zahalak (1972), where it was found that, as $\theta\to\theta_0$,

$$\left.\begin{aligned}&A(\theta) = -\frac{2^{\frac{1}{2}}C_0}{\pi(\theta_0-\theta)}+o\left(\frac{1}{\theta_0-\theta}\right),\\ &B(\theta) = C(\theta) = D(\theta) = 0 \qquad \text{for}\quad \theta>\theta_0,\\ &\left.\begin{aligned}B(\theta) &= 2^{\frac{1}{2}}v_0'(r_0)/(\beta\cos^2\theta_0)+o(1)\\ C(\theta) &= C_0+o(1)\\ D(\theta) &= -\tfrac{1}{2}C_0(\theta_0-\theta)^2+o((\theta_0-\theta)^2)\end{aligned}\right\} \quad \text{for}\quad \theta<\theta_0,\\ &\quad C_0 = v_0(r_0)/\beta\sin\theta_0.\end{aligned}\right\} \qquad (2.11)$$

From (2.11) and (2.8) it is clear that $w^{(1)}(0,\theta_0^-)$ in (2.9) is equal to $-C_0$.

One could attempt to construct further terms in the outer expansion (2.1). But, as Lighthill (1949*b*) pointed out, the fact that $f_{rr} = O(|1-r|^{-\frac{1}{2}})$ on $r = 1$ will cause successive terms in this expansion to be progressively more singular on $r = 1$, indicating that the outer expansion is not uniformly valid in regions containing the unit circle. In addition, of course, the outer expansion is not uniformly valid near the plane characteristic AB in figure 2. In §§ 3 and 4 the leading terms of inner expansions, that correct these deficiencies of the outer expansion, will be considered.

3. Inner expansions away from the singular point

Construction of inner expansions to represent the solution near the wave fronts, where the outer solution is not valid, is considered in detail by Zahalak 1972). These calculations are lengthy, but represent a straightforward application of the methods presented by Whitham (1956) and Van Dyke (1964). Except

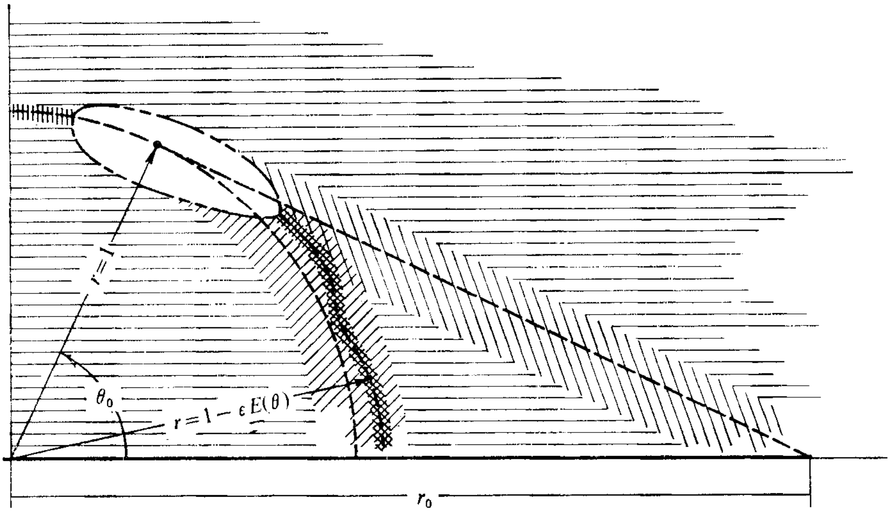

FIGURE 3. Regions of validity of various expansions (schematic). - - - -, I (inner expansion); ▥, I_L; ▩, I_R; ⊟, O (outer expansion); ▧, I_M; ▨, M_R.

for one illustrative case, details are omitted for the sake of brevity, and only the final results are presented. It is found that four separate 'inner' expansions are required to describe the flow in regions contiguous to the singular point. These expansions are denoted by the symbols I_L, I_M, M_R and I_R, and their regions of validity are schematically indicated in figure 3. In each case below, the results may be verified by substituting the indicated expansions and inner variables into the equations of motion (1.3), and finding the limiting form of the equations as $\epsilon \to 0$ with the inner variables fixed. It is found that, to first order in each of the expansions, the relations $p = \rho = -M^2 w$ hold. Then each of the results below is an exact solution to the corresponding system of inner equations, and each satisfies the matching conditions of Van Dyke (1964).

Near the diffracted front to the left of the singular point, the I_L expansion holds. The inner variables in this region are defined as

$$\hat{X} = \epsilon^{-2}(1-r) \quad \text{and} \quad \hat{Y} = \theta_0 - \theta.$$

A sketch of the analysis for this case follows; the other cases are treated in an essentially similar manner. Gauge analysis indicates that the I_L expansion has the form

$$\left.\begin{gathered} u = \epsilon^2 \hat{U}^{(1)}(\hat{X}, \hat{Y}) + o(\epsilon^2), \quad v = \epsilon^4 \hat{V}^{(1)}(\hat{X}, \hat{Y}) + o(\epsilon^4), \quad w = \epsilon^2 \hat{W}^{(1)}(\hat{X}, \hat{Y}) + o(\epsilon^2), \\ p = \epsilon^2 \hat{P}^{(1)}(\hat{X}, \hat{Y}) + o(\epsilon^2), \quad \rho = \epsilon^2 \hat{R}^{(1)}(\hat{X}, \hat{Y}) + o(\epsilon^2). \end{gathered}\right\} \tag{3.1}$$

If (3.1) is substituted directly into the equations of motion (1.3), the inner limit yields the following system of first-order inner equations:

$$-\beta \hat{U}^{(1)}_{\hat{X}} + \hat{W}^{(1)}_{\hat{X}} + \hat{R}^{(1)}_{\hat{X}} = 0, \tag{3.2a}$$

$$M^2 \hat{U}^{(1)}_{\hat{X}} - \beta \hat{P}^{(1)}_{\hat{X}} = 0, \quad M^2 \hat{V}^{(1)}_{\hat{X}} - \beta \hat{P}^{(1)}_{\hat{Y}} = 0, \tag{3.2b,c}$$

$$M^2 \hat{W}^{(1)}_{\hat{X}} + \hat{P}^{(1)}_{\hat{X}} = 0, \quad M^2(\gamma - 1) \hat{W}^{(1)}_{\hat{X}} + \gamma \hat{P}^{(1)}_{\hat{X}} - \hat{R}^{(1)}_{\hat{X}} = 0. \tag{3.2d,e}$$

Inspection of the system (3.2) shows that it is insufficient to determine the five quantities $\hat{U}^{(1)}$, $\hat{V}^{(1)}$, $\hat{W}^{(1)}$, $\hat{P}^{(1)}$ and $\hat{R}^{(1)}$, because the equations are not all independent: the linear combination

$$(3.2e)+(\beta^2/M^2-\gamma)(3.2d)+(\beta^2/M^2)(3.2b)+(3.2a) \tag{3.3}$$

vanishes identically. This difficulty may be overcome in various ways. For example, the equations of motion could be expanded to second order, and a fifth independent relation between the first-order coefficients could be obtained from the second-order terms in this expansion. However, this approach requires a knowledge of the I_L expansion to two terms. As an alternative, we have chosen to replace the last equation of (1.3) by the linear combination (3.3) of the basic field equations (1.3). The inner limit of this new equation yields, after simplification,

$$\{2\hat{X}+(M^2/\beta^2)(2(\beta\hat{U}^{(1)}-\hat{W}^{(1)})+\gamma\hat{P}^{(1)}-\hat{R}^{(1)})\}\,\hat{U}^{(1)}_{\hat{X}}-\hat{U}^{(1)}=0, \tag{3.4}$$

a fifth independent equation for the determination of the five first-order coefficients. Similar problems arise in expanding the shock conditions, and these may be handled in a similar way. This simple system of first-order inner equations may be readily integrated to yield (after imposition of the shock and matching conditions)

$$\left.\begin{aligned} &\hat{U}^{(1)}=\hat{V}^{(1)}=\hat{W}^{(1)}=0 \quad \text{for} \quad \hat{X}<\hat{X}_d(\hat{Y}),\\ &\left.\begin{aligned}\hat{U}^{(1)}&=\beta A\{[(\beta KA)^2+\hat{X}]^{\frac{1}{2}}+\beta KA\}\\ \hat{W}^{(1)}&=-A\{[(\beta KA)^2+\hat{X}]^{\frac{1}{2}}+\beta KA\}\\ V^{(1)}&=\tfrac{2}{3}\beta\frac{dA}{d\hat{Y}}\{[(\beta KA)^2+\hat{X}]^{\frac{1}{2}}+\beta KA\}^3\end{aligned}\right\}\quad \text{for}\quad \hat{X}>\hat{X}_d(\hat{Y}),\end{aligned}\right\} \tag{3.5}$$

where $K=\frac{1}{2}(\gamma+1)M^4\beta^{-3}$. (The functions A, B, C, D are the expansion coefficients defined in §2.) If $A(\hat{Y})<0$, $\hat{X}_d(\hat{Y})=0$ represents a characteristic of discontinuity, while, if $A(\hat{Y})>0$, there is a shock given by

$$\hat{X}_d(\hat{Y})=-\tfrac{3}{4}(\beta KA)^2+\ldots. \tag{3.6}$$

Near the main front to the right of the singular point, inner variables are chosen as $\tilde{X}=\epsilon^{-1}\{1-r\cos(\theta_0-\theta)\}$ and $\tilde{Y}=\theta_0-\theta$. Then in the I_M expansion we have $u=v=w=0$ for $\tilde{X}<0$, while for $\tilde{x}>\tilde{X}_d(\tilde{Y})$,

$$(u,v,w)=\epsilon(v_0(r_0)/\sin\theta_0)(\cos\tilde{Y},\sin\tilde{Y},-\beta^{-1})+\ldots. \tag{3.7}$$

If $v_0(r_0)<0$,

$$\tilde{X}_d(\tilde{Y})=-\{Kv_0(r_0)\sin(\theta_0-\tilde{Y})\}\{\cos\tilde{Y}\sin^2\theta_0\}^{-1}+\ldots$$

represents a characteristic of discontinuity, while, if $v_0(r_0)>0$, there is a shock given by $\tilde{X}_d(\tilde{Y})=-\{Kv_0(r_0)\sin(\theta_0-\tilde{Y})\}\ \{2\cos\tilde{Y}\sin^2\theta_0\}^{-1}+\ldots.$ In the latter case, there is an expansion wave in the region $0<\tilde{X}<\tilde{X}_d(Y)$ given by

$$(u,v,w)=\epsilon\tilde{X}\{\sin\theta_0\cos\tilde{Y}\}\{K\sin(\theta_0-\tilde{Y})\}^{-1}(\cos\tilde{Y},\sin\tilde{Y},-\beta^{-1}). \tag{3.8}$$

To correct the non-uniform validity of the outer expansion near the diffracted front to the right of the singular point, the 'middle' variables $\bar{X}=\epsilon^{-1}(1-r)$ and

$\bar{Y} = \theta_0 - \theta$ are first introduced. Solution of the first-order problem yields the leading term in the M_R expansion

$$u = \epsilon\beta C + \ldots v = -\epsilon\beta(dD/d\bar{Y}) + \ldots w = \epsilon(D-C) + \ldots. \tag{3.9}$$

However, if the next term in this expansion is computed, one finds, for example, that

$$u = \epsilon\beta C + \epsilon^{\frac{3}{2}}\beta \begin{cases} B[E-\bar{X}]^{\frac{1}{2}} & (\bar{X} < E) \\ A[\bar{X}-E]^{\frac{1}{2}} & (\bar{X} > E) \end{cases} + \ldots, \tag{3.10}$$

where
$$E(\theta) = -\beta K C(\theta) + M^2 2^{-1}\beta^{-2}\{2 + (\gamma-1)M^2\} D(\theta). \tag{3.11}$$

Thus, the M_R expansion shifts the square-root singularity of the linearized solution from $r = 1$ to $r = 1 - \epsilon E(\theta)$, which is the first-order location of the characteristic of discontinuity or shock found by Lighthill (1949*b*). For reasons analogous to those discussed by Lighthill (1949*b*), it is expected that the singularity in the second term of the M_R expansion will render this expansion invalid sufficiently close to $\bar{X} = E(\bar{Y})$, and a supplementary inner expansion I_R is required to describe the flow in this region.

For the I_R expansion, the inner variables are defined as $\bar{\bar{X}} = \epsilon^{-1}\{(1-r) - \epsilon E(\theta)\}$ and $\bar{\bar{Y}} = \theta_0 - \theta$. Solution of the first-order I_R problem yields

$$u = \epsilon\beta C + \ldots, \quad v = -\epsilon\beta\frac{dD}{d\bar{\bar{Y}}} + \ldots, \quad w = \epsilon(D-C) + \ldots. \tag{3.12}$$

This is identical to the first term of the M_R expansion; the differences between M_R and I_R first appear in the second terms of these expansions. While the form of these second-order terms can be found by solving the appropriate second-order equations, it appears that not all of the arbitrary functions of integration entering these solutions can be determined by matching unless the second term of the outer expansion is available. Therefore, for the purposes of this study, only the leading terms in each expansion are listed. However, the first approximation to the shock *strength* may be found from the second term of the I_R expansion (which is $O(\epsilon^2)$ in p) regardless of the arbitrary functions of integration. This is a situation analogous to that found by Lighthill (1949*b*), and in fact the shock strength so computed agrees exactly with that given by Lighthill. The form of an approximate solution constructed by the method of Whitham (1956) for the region near the diffracted front to the right of the singular point clearly indicates that two separate expansions are required to describe the solution.

The behaviour of $A(\theta)$, $B(\theta)$, etc. (listed at the end of §2) as $\theta \to \theta_0$, and the results of this section, lead to the following conclusions.

(i) Sufficiently close to the singular point two types of behaviour are possible. If $v_0(r_0) > 0$, there are shocks near the main front and the diffracted front to the left of the singular point, while there is an expansion across a characteristic of discontinuity near the diffracted front to the right of the singular point. If $v_0(r_0) < 0$, the situation is reversed: there are expansions across characteristics of discontinuity near the main front, and diffracted front to the left of the singular point, while there is a shock near the diffracted front to the right of the singular point.

(ii) These inner expansions, together with the outer expansion, cannot correctly represent the flow near the singular point, since the strength of the shock near the diffracted front approaches infinity as $\theta \to \theta_0$.

In §4 we consider the leading term of an inner expansion describing the flow in the vicinity of the singular point.

4. The inner expansion near the singular point

4.1. *Formulation of the first-order inner problem*

It is clear that the inner expansions listed in §3 to represent the solution near the diffracted front cannot be valid near the singular point, because they indicate that both the shock position $\eta(\theta)$ and the shock strength approach infinity as $\theta \to \theta_0$. A separate inner expansion is needed to represent the solution near the singular point. An examination of the contiguous expansions indicates that an appropriate definition for the inner variables is

$$X = (1-r)/\epsilon \quad \text{and} \quad Y = (\theta_0 - \theta)/\epsilon^{\frac{1}{2}}, \tag{4.1}$$

and, with the inner variables so defined, the inner expansion takes the form

$$q(r,\theta;\epsilon) = Q(X,Y;\epsilon) = \Delta_1(\epsilon)\,Q^{(1)}(X,Y) + \Delta_2(\epsilon)\,Q^{(2)}(X,Y) + \ldots, \tag{4.2}$$

where $$Q^{(i)T} = [U^{(i)}, V^{(i)}, W^{(i)}, P^{(i)}, R^{(i)}],$$

the Δ_i are diagonal matrices of gauge functions as described in §2, and

$$\Delta_{11} = \Delta_{13} = \Delta_{14} = \Delta_{15} = \epsilon,$$

while $\Delta_{12} = \epsilon^{\frac{3}{2}}$. When this expansion is inserted into the field equations (1.3) and the inner limit taken, there results the system of five equations for the determination of $q^{(1)}$†

$$\left.\begin{aligned}
&-U^{(1)}_X + \frac{1}{\beta}(W^{(1)}_X + R^{(1)}_X) = 0,\\
&\frac{M^2}{\beta}U^{(1)}_X - P^{(1)}_X = 0, \quad \frac{M^2}{\beta}V^{(1)}_X - P^{(1)}_Y = 0, \quad M^2 W^{(1)}_X + P^{(1)}_X = 0,\\
&\left\{2X + \frac{M^2}{\beta^2}(2(\beta U^{(1)} - W^{(1)}) + \gamma P^{(1)} - R^{(1)})\right\} U^{(1)}_X - U^{(1)} + V^{(1)}_Y = 0.
\end{aligned}\right\} \tag{4.3}$$

If shocks or characteristics of discontinuity are assumed to occur in the inner region, the appropriate first-order characteristic equations and shock conditions may be obtained by expanding (1.4) and (1.5) under the assumption

$$\left.\begin{aligned}
(1-r) &= h(\theta;\epsilon) = h^*(Y;\epsilon) = \epsilon h^{(1)*}(Y) + o(\epsilon) \text{ on a characteristic,}\\
(1-r) &= \eta(\theta;\epsilon) = \eta^*(Y;\epsilon) = \epsilon\eta^{(1)*}(Y) + o(\epsilon) \text{ on a shock.}
\end{aligned}\right\} \tag{4.4}$$

For the characteristic this yields

$$\frac{dh^{(1)*}}{dY} = \left(\frac{dX}{dY}\right)_{\text{char.}} = \pm(-S)^{-\frac{1}{2}}, \tag{4.5}$$

where $$S = 2X + M^2\beta^{-2}\{2(\beta U^{(1)} - W^{(1)}) + \gamma P^{(1)} - R^{(1)}\}.$$

† See remarks preceding (3.4).

If the $U^{(1)}$, $W^{(1)}$, etc., are bounded, then real characteristics will exist as $X \to -\infty$ and will not exist as $X \to +\infty$, which corresponds to the fact that the equations governing the first-order outer flow are hyperbolic outside the unit circle, and elliptic inside.

Instead of inserting (4.4) into (1.6) and expanding, a completely equivalent system of first-order shock conditions may be obtained more directly by regarding (4.3) as a system of conservation laws, and by constructing the discontinuity conditions which weak solution of this system must satisfy. Writing (4.3) in divergence form,

$$\begin{bmatrix} R^{(1)} - \dfrac{M^2}{\beta} U^{(1)} \\ P^{(1)} - \dfrac{M^2}{\beta} U^{(1)} \\ W^{(1)} + \dfrac{1}{\beta} U^{(1)} \\ \dfrac{M^2}{\beta} V^{(1)} \\ (S - KU^{(1)})\, U^{(1)} \end{bmatrix}_X + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -P^{(1)} \\ V^{(1)} \end{bmatrix}_Y + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -3U \end{bmatrix} = 0,$$

and denoting a shock surface by $\omega(X, Y) = 0$, the shock conditions follow immediately as (Courant & Hilbert 1966, p. 489)

$$\left.\begin{aligned} &\left[R^{(1)} - \frac{M^2}{\beta} U^{(1)}\right] \omega_X = 0, \quad \left[P^{(1)} - \frac{M^2}{\beta} U^{(1)}\right] \omega_X = 0, \\ &\left[W^{(1)} + \frac{1}{\beta} U^{(1)}\right] \omega_X = 0, \quad \left[\frac{M^2}{\beta} V^{(1)}\right] \omega_X - [P^{(1)}]\, \omega_Y = 0, \\ &[(S - KU^{(1)})\, U^{(1)}]\, \omega_X + [V^{(1)}]\, \omega_Y = 0. \end{aligned}\right\} \tag{4.6}$$

If $\omega_X = 0$, then the surface of discontinuity is given by $Y = \text{constant}$ and the first three conditions are automatically satisfied, while the last two require

$$[P^{(1)}] = [V^{(1)}] = 0,$$

i.e. the pressure and normal velocity must be continuous across the surface. This represents a contact discontinuity. The discontinuities in $U^{(1)}$, $W^{(1)}$ and $R^{(1)}$ appear to be arbitrary, but it will be shown shortly that these functions are all proportional to $P^{(1)}$, and therefore must also be continuous. It does not appear possible to have contact discontinuities in the inner solution. If $\omega_X \neq 0$, then

$$\omega_Y / \omega_X = -dX/dY$$

on the shock, and (4.6) may be rewritten in an obvious manner in terms of dX/dY.

In addition to (4.3), and shock conditions (4.6), the inner solution must satisfy matching conditions with the outer solution of § 2 and all the contiguous inner solutions of § 3. It is assumed that the shocks and characteristics of discontinuity separating the disturbed from the undisturbed flow occur in the inner region, and therefore $Q^{(1)}$ vanishes identically for $X < X_d(Y)$, where $X_d(Y)$ is the locus of these discontinuities.

The first, second and fourth equations of (4.3) may be rearranged to yield

$$\left(W^{(1)}+\frac{1}{\beta}U^{(1)}\right)_X = 0, \quad \left(P^{(1)}-\frac{M^2}{\beta}U^{(1)}\right)_X = 0, \quad \left(R^{(1)}-\frac{M^2}{\beta}U^{(1)}\right)_X = 0.$$

In any region free of shocks, the general solution of these equations is

$$W^{(1)}+\frac{1}{\beta}U^{(1)} = g_1(Y), \quad P^{(1)}-\frac{M^2}{\beta}U^{(1)} = g_2(Y), \quad R^{(1)}-\frac{M^2}{\beta}U^{(1)} = g_3(Y).$$

The first three shock conditions (4.6) (assuming $\omega_X \neq 0$) require that g_1, g_2 and g_3 be continuous across any shock. Therefore, g_1, g_2 and g_3 are continuous everywhere. But, since $Q^{(1)} \equiv 0$ for X sufficiently negative, we must have

$$g_1(Y) = g_2(Y) = g_3(Y) = 0,$$

from which

$$W^{(1)} = -\frac{1}{\beta}U^{(1)}, \quad P^{(1)} = \frac{M^2}{\beta}U^{(1)}, \quad R^{(1)} = \frac{M^2}{\beta}U^{(1)} \tag{4.7}$$

everywhere. Using (4.7), the third and fifth equations of (4.3) become

$$2(X+KU^{(1)})\,U_X^{(1)} - U^{(1)} + V_Y^{(1)} = 0 \quad \text{and} \quad V_X^{(1)} - U_Y^{(1)} = 0, \tag{4.8}$$

while the last two shock conditions (4.6), after some rearrangement, may be written as

$$[V^{(1)}]+[U^{(1)}]\frac{dX}{dY} = 0 \quad \text{and} \quad \left(\frac{dX}{dY}\right)^2 + 2X = -K\{[U^{(1)}]+2U_1^{(1)}\} \tag{4.9}$$

on the shock, where $U_1^{(1)}$ is evaluated on the front side of the shock.

The system (4.8) furnishes two equations for $U^{(1)}$ and $V^{(1)}$, which, together with shock and matching conditions, are assumed uniquely to determine these functions. $W^{(1)}$, $P^{(1)}$, $R^{(1)}$ are then completely determined by (4.7). If we define

$$x = \frac{X}{\beta|C_0|K}, \quad y = \frac{Y}{(\beta|C_0|K)^{\frac{1}{2}}}, \quad \xi = \frac{U^{(1)}}{B|C_0|}, \quad \eta = \frac{V^{(1)}}{(\beta|C_0|)^{\frac{3}{2}}K^{\frac{1}{2}}}, \tag{4.10}$$

where $C_0 = C(\theta_0)$ (see §2 for the definition of $C(\theta)$), then (4.8) and (4.9) become respectively

$$2(x+\xi)\,\xi_x - \xi + \eta_y = 0 \quad \text{and} \quad \eta_x - \xi_y = 0, \tag{4.11}$$

and, on the shock,

$$\frac{dx}{dy} = -\frac{[\eta]}{[\xi]} \quad \text{and} \quad \left(\frac{dx}{dy}\right)^2 + 2x = -\{[\xi]+2\xi_1\}. \tag{4.12}$$

The second equation of (4.11) implies that the problem may be posed in terms of a potential:

$$2(x+\Phi_x)\,\Phi_{xx} - \Phi_x + \Phi_{yy} = 0, \tag{4.11a}$$

where $\xi = \Phi_x$ and $\eta = \Phi_y$. Equation (4.11*a*) was obtained by Kuo (1955) by more informal methods, starting with the assumption of irrotational flow, and the first equation of (4.12) is equivalent to his condition of continuity of Φ across the shock. It will be found convenient for certain purposes to examine the solutions of (4.11) in the hodograph (ξ, η) plane. Introducing the hodograph transformation, (4.11) becomes

$$2(x+\xi)\,y_\eta + x_\xi - j\xi = 0 \quad \text{and} \quad x_\eta - y_\xi = 0, \tag{4.13}$$

where $j = x_\xi y_\eta - x_\eta y_\xi$. This transformation is valid at all points where neither j nor j^{-1} vanishes.

Before proceeding with the construction of particular solutions of (4.11), we obtain the inner expansions of some of the contiguous solutions given in §§2 and 3. These expansions provide constraints on the inner expansion, and furnish a guide in constructing particular solutions of the inner equations. In §3 we introduced a notation to identify the various inner expansions valid near the wave fronts, but away from the singular point. We now further introduce the symbol O to denote the outer expansion, and the symbol I to denote the inner expansion valid near the singular point. To symbolize the expansion obtained at various steps in the matching process, we introduce the shorthand $(m-I_j)/(n-I_k)$ to mean: the m-term I_j expansion of the n-term I_k expansion. With these definitions, and the results of §3, we have the following.

The $(1-I)/(1-I_L)$ *expansion.*

$$X_d = \epsilon \hat{X}_d = \begin{cases} 0 & (v_0(r_0) < 0), \\ -\dfrac{3\beta^2 K^2 C_0^2}{2\pi^2}\dfrac{1}{Y^2} + \dots & (v_0(r_0) > 0), \end{cases}$$

$$u = \epsilon^2 \hat{U}^{(1)} + \dots = \epsilon^2 \left\{ -\frac{\beta \times 2^{\frac{1}{2}} C_0}{\pi \epsilon^{\frac{1}{2}} Y} \left(\left[\left(\frac{\beta K \times 2^{\frac{1}{2}} C_0}{\pi \epsilon^{\frac{1}{2}} Y} \right)^2 + \frac{X}{\epsilon} \right]^{\frac{1}{2}} - \frac{\beta K \times 2^{\frac{1}{2}} C_0}{\pi Y \epsilon^{\frac{1}{2}}} \right) + \dots \right\} + \dots,$$

$$v = \epsilon^4 \hat{V}^{(1)} + \dots = \epsilon^4 \left\{ \frac{2^{\frac{3}{2}} \beta C_0}{3\pi \epsilon Y^2} \left(\left[\left(\frac{\beta K \times 2^{\frac{1}{2}} C_0}{\pi \epsilon^{\frac{1}{2}} Y} \right)^2 + \frac{X}{\epsilon} \right]^{\frac{1}{2}} - \frac{\beta K \times 2^{\frac{1}{2}} C_0}{\pi \epsilon^{\frac{1}{2}} Y} \right)^3 + \dots \right\} + \dots,$$

for $X > X_d$, while u and v both vanish for $X < X_d$. In terms of the x and y of (4.10), these results may be written

$$\left. \begin{aligned} \frac{u}{\beta |C_0|} &= \epsilon \left\{ \mp \frac{\sqrt{2}}{\pi y} \left(\left[\frac{2}{\pi^2 y^2} + x \right]^{\frac{1}{2}} \mp \frac{\sqrt{2}}{\pi y} \right) \right\} + \dots, \\ \frac{v}{(\beta |C_0|)^{\frac{3}{2}} K^{\frac{1}{2}}} &= \epsilon^{\frac{3}{2}} \left\{ \pm \frac{2^{\frac{3}{2}}}{3\pi y^2} \left(\left[\frac{2}{\pi^2 y^2} + x \right]^{\frac{1}{2}} \mp \frac{\sqrt{2}}{\pi y} \right)^3 \right\} + \dots \end{aligned} \right\} \tag{4.14}$$

for $x < x_d(y)$, where

$$x_d = \begin{cases} 0 & (v_0(r_0) < 0), \\ -\dfrac{3}{2\pi^2 y^2} + \dots & (v_0(r_0) > 0). \end{cases}$$

In (4.14) the upper signs are chosen if $v_0(r_0) > 0$ and the lower signs are chosen if $v_0(r_0) < 0$. This convention will be used for the remainder of §4.

The $(1-I)/(1-0)$ *expansion.*

$$-s = \operatorname{sech}^{-1}(r) = (2(1-r))^{\frac{1}{2}} + O(1-r)^{\frac{3}{2}} = (\epsilon 2X)^{\frac{1}{2}} + O(\epsilon^{\frac{3}{2}}).$$

Using (2.10) and (2.11) it is easy to show that (Zahalak 1972)

$$\left.\begin{aligned} \frac{u}{\beta|C_0|} &= \pm\epsilon F\left(\frac{y}{(2x)^{\frac{1}{2}}}\right)+\dots,\\ \frac{v}{(\beta|C_0|)^{\frac{3}{2}}K^{\frac{1}{2}}} &= \pm\epsilon^{\frac{3}{2}}(2x)^{\frac{1}{2}}\left\{\frac{1}{\pi}+\frac{y}{(2x)^{\frac{1}{2}}}F\left(\frac{y}{(2x)^{\frac{1}{2}}}\right)\right\}+\dots,\end{aligned}\right\} \tag{4.15}$$

where

$$F(t) = \tfrac{1}{2}+\frac{1}{\pi}\tan^{-1}(t). \tag{4.16}$$

Equation (4.15) can also be obtained as the first term of the inner expansion of the uniform wave-front expansions discussed in Myers (1971).

In matching (4.15) with the inner expansion, we must take the limit of

$$\frac{u}{(\beta|C_0|)} = \epsilon\xi\left[\frac{1-r}{\epsilon\beta K|C_0|},\ \frac{\theta_0-\theta}{(\epsilon\beta K|C_0|)^{\frac{1}{2}}}\right]+\dots,$$

$$\frac{v}{(\beta|C_0|)^{\frac{3}{2}}K^{\frac{1}{2}}} = \epsilon^{\frac{3}{2}}\eta\left[\frac{1-r}{\epsilon\beta K|C_0|},\ \frac{\theta_0-\theta}{(\epsilon\beta K|C_0|)^{\frac{1}{2}}}\right]+\dots,$$

as $\epsilon\to 0$, with r and θ held fixed, which is the same as obtaining the limiting behaviour of $\xi(x,y)$ and $\overline{\eta}(x,y)$ as $x\to\infty$ with $y/(2x)^{\frac{1}{2}}$ held fixed. Equation (4.15) shows that, in this limit, we must have

$$\xi = \pm F\left(\frac{y}{(2x)^{\frac{1}{2}}}\right)+\dots \quad\text{and}\quad \eta = \pm(2x)^{\frac{1}{2}}\left\{\frac{1}{\pi}+\left(\frac{y}{(2x)^{\frac{1}{2}}}\right)F\left(\frac{y}{(2x)^{\frac{1}{2}}}\right)\right\}+\dots. \tag{4.17}$$

The above conditions may be inverted to obtain the form of this matching condition in the hodograph plane, which will be used in § 5. Equation (4.17) shows that ξ approaches a constant (positive for $v_0(r_0) > 0$ and negative for $v_0(r_0) < 0$, since $F > 0$), while $\eta\to\pm\infty$. Inverting (4.17),

$$x = \frac{\eta^2}{2\{(1/\pi)-\xi\cot(\pi\xi)\}^2}+\dots,\quad y = \frac{-\eta\cot(\pi\xi)}{\{(1/\pi)-\xi\cot(\pi\xi)\}}+\dots. \tag{4.18}$$

Therefore, in the hodograph plane, the matching condition between the one-term inner and outer expansions requires that $x(\xi,\eta)$ and $y(\xi,\eta)$ have the asymptotic behaviour (4.18) as $\eta\to\pm\infty$ on $\xi=$ constant. The expressions (4.18) are valid for both $v_0(r_0) < 0$ and $v_0(r_0) > 0$, and the difference lies in the fact that for $v_0(r_0) > 0$ the hodograph image of the motion lies in the first quadrant $u > 0$, $v > 0$, while for $v_0(r_0) < 0$ the image lies in the third quadrant $u < 0$, $v < 0$. If we define

$$\phi(\xi) = \frac{\pi}{\pi\xi-\tan(\pi\xi)}, \tag{4.19}$$

the matching condition may be written as

$$x = \tfrac{1}{2}\phi'(\xi)\,\eta^2+\dots,\quad y = \phi(\xi)\,\eta+\dots, \tag{4.20}$$

as $\eta\to\pm\infty$ on $\xi=$ constant.

4.2. *Particular solutions*

Both (4.11) and (4.13) are nonlinear, and there is no analytical procedure known for constructing solutions satisfying prescribed auxiliary conditions. However, it is feasible to seek particular solutions by trial and error. First, $\xi \equiv \eta \equiv 0$ is a solution, and this solution is appropriate ahead of the curve (shock and/or characteristic) which forms the boundary of the disturbed region. Next, the first term of the $(1-I)/(I-I_M)$ expansion, which can easily be obtained for $v_0(r_0) > 0$ as

$$\xi = 1, \quad \eta = y \quad \text{for} \quad x+\tfrac{1}{2}y^2 > -\tfrac{1}{2},$$

and for $v_0(r_0) < 0$ as

$$\xi = \begin{cases} -(x+\tfrac{1}{2}y^2), & 0 < (x+\tfrac{1}{2}y^2) < 1 \\ -1, & (x+\tfrac{1}{2}y^2) > 1 \end{cases}, \quad \eta = \begin{cases} -y(x+\tfrac{1}{2}y^2), & 0 < (x+\tfrac{1}{2}y^2) < 1 \\ -y, & (x+\tfrac{1}{2}y^2) > 1 \end{cases}, \tag{4.21}$$

satisfies (4.11) exactly, indicating that the expansion near the main front is valid into the inner region. Equations (4.11) are hyperbolic with respect to the solutions $\xi = \pm 1$, $\eta = \pm y$ for $x < \mp 1$, and elliptic below this line, which is itself a characteristic of (4.11). It is clear that the solution $\xi = \pm 1$, $\eta = \pm y$ cannot extend into $x > \mp 1$, since an examination of the M_R and I_R expansions shows that, for $v_0(r_0) > 0$, $x = -1$ is the inner limit of the characteristic of discontinuity, while for $v_0(r_0) < 0$, a shock stands ahead of $x = +1$.

To obtain a solution representing the remainder of the field, we use the hodograph equations (4.13). The form of the matching conditions (4.20) suggests that we seek a solution in the form of a polynomial:

$$x = \eta^2\Psi_2^*(\xi)+\eta\Psi_1^*(\xi)+\Psi_0^*(\xi), \quad y = \eta\Psi_1(\xi)+\Psi_0(\xi). \tag{4.22}$$

The second equation of (4.13) is satisfied if

$$\Psi_2^* = \tfrac{1}{2}\Psi_1' \quad \text{and} \quad \Psi_1^* = \Psi_0'.$$

Substituting into the first equation of (4.13), and ordering by powers of η, we obtain

$$\eta^2\{(1-\xi\Psi_1)\Psi_1''+2(\Psi_1+\xi\Psi_1')\Psi_1'\}+\eta\{(1-\xi\Psi_1)\Psi_0''+2(\Psi_1+\xi\Psi_1')\Psi_0'\} \\ +\{(1-\xi\Psi_1)\Psi_0^{*\prime}+2\Psi_1\Psi_0^*+\xi(2\Psi_1+\Psi_0'^2)\} = 0. \tag{4.23}$$

The quantities in each of the three brackets are set equal to zero, which yields three ordinary differential equations for the three functions Ψ_1, Ψ_0 and Ψ_0^*. The first equation is nonlinear, but may be integrated in closed form. However, if the matching condition (4.20) is to be satisfied, we must have $\Psi_1(\xi) = \phi(\xi)$. Therefore, we only need to verify that $\phi(\xi)$ is a particular solution of the first differential equation, which can be done easily. The second and third equations are linear in the unknown functions, and have the general solution

$$\Psi_0(\xi) = k_1\phi(\xi)+k_2, \quad \Psi_0^*(\xi) = \tfrac{1}{2}k_1^2\phi'(\xi)-g(\xi;k_3),$$

where

$$g(\xi;k_3) = \xi+\frac{1}{2\pi}\sin(2\pi\xi)+k_3\sin^2(\pi\xi), \tag{4.24}$$

and k_1, k_2 and k_3 are arbitrary constants. The solution (4.24) is now completely determined, and can be written in the more compact form

$$x = \tfrac{1}{2}(\eta-\eta_0)^2\,\phi'(\xi) - g(\xi;k_3), \quad y-y_0 = (\eta-\eta_0)\,\phi(\xi), \tag{4.25}$$

where η_0 and y_0 are arbitrary constants. Equation (4.25) can also be written in the form

$$\xi = \pm F(\alpha), \quad (\eta-\eta_0) = \pm(y-y_0)\left\{\frac{1}{\pi\alpha}+F(\alpha)\right\}, \tag{4.26}$$

where $\alpha(x,y)$ is defined implicitly by

$$x - \frac{(y-y_0)^2}{2\alpha^2} + g^*(\alpha) = 0, \tag{4.27}$$

and

$$g^*(\alpha) = g[\pm F(\alpha)] = \pm F(\alpha) \mp \frac{\alpha}{\pi(1+\alpha^2)} + \frac{k_3}{1+\alpha^2}. \tag{4.28}$$

Equation (4.27) shows that the curves of constant α, and therefore the curves of constant ξ are parabolas whose axis of symmetry is $y = y_0$. Such curves are shown, for $y_0 = \eta_0 = k_3 = 0$, $v_0(r_0) > 0$, in figure 5. The solution is a many-valued function of the variables x, y in a region bounded by the envelope of the curves $\alpha =$ constant and $x = \frac{1}{2} \mp \frac{1}{2}$. There is a branch point at $x = \mp\frac{1}{2} - k_3$, $y = y_0$. α takes on all real values, and $\alpha = -\infty$ corresponds to $x = 0$, $y < y_0$, while $\alpha = +\infty$ corresponds to $x = \mp 1$, $y > y_0$. The equation of the envelope may be written as

$$x_e = -\left\{\frac{\alpha}{2}\frac{dg^*}{d\alpha} + g^*\right\}, \quad y_e = y_0 \mp \left(\pm\frac{dg^*}{d\alpha}\right)^{\frac{1}{2}}(\mp\alpha)^{\frac{3}{2}}.$$

4.3. *Matching*

We first compute $(1-I_L)/(1-I)$:

$$u = \epsilon U^{(1)}(X, Y) + \ldots = \epsilon U^{(1)}\left(\epsilon\hat{X}, \frac{\hat{Y}}{\epsilon^{\frac{1}{2}}}\right) + \ldots \quad \text{with} \quad \hat{Y} < 0.$$

We require the limit of $U^{(1)}$ as $\epsilon \to 0$ with $\hat{X}$ and $\hat{Y}$ fixed, which is the same as the limit of $U^{(1)}(x,y)$ as $x \to 0$, $y \to -\infty$ with (xy^2) fixed. From (4.27) it is clear that $\alpha \to -\infty$ in this limit. As $\alpha \to -\infty$,

$$x - \frac{(y-y_0)^2}{2\alpha^2} + \left(\mp\frac{2}{\pi\alpha} + \ldots\right) = 0.$$

Retaining only the leading term in the parentheses, and solving the resulting quadratic equation on α^{-1}:

$$\begin{aligned}\frac{1}{\pi\alpha} &= \mp\frac{2}{\pi^2(y-y_0)^2}\{1 \pm [1+\tfrac{1}{2}\pi^2(y-y_0)^2 x]^{\frac{1}{2}}\} + \ldots \\ &= \mp\frac{2}{\pi^2 y^2}\{1 \pm [1+\tfrac{1}{2}\pi^2 y^2 x]^{\frac{1}{2}}\} + \ldots.\end{aligned} \tag{4.29}$$

Expanding,

$$\begin{aligned}\frac{u}{\beta|C_0|} &= \epsilon\xi + \ldots = \epsilon F(\alpha) + \ldots = \pm\epsilon\left\{-\frac{1}{\pi\alpha} + \frac{1}{3\pi\alpha^3} + \ldots\right\} \\ &= \epsilon\frac{2}{\pi^2 y^2}\{1 \pm [1+\tfrac{1}{2}\pi^2 y^2 x]^{\frac{1}{2}}\} + \ldots,\end{aligned} \tag{4.30}$$

$$\frac{v}{(\beta|C_0|)^{\frac{3}{2}} K^{\frac{1}{2}}} = \epsilon^{\frac{3}{2}}\eta + \ldots = \epsilon^{\frac{3}{2}}\left\{\eta_0 \pm (y-y_0)\left(\frac{1}{\pi\alpha} - \frac{1}{\pi\alpha} + \frac{1}{3\pi\alpha^3} + \ldots\right)\right\} + \ldots$$

$$= \epsilon^{\frac{3}{2}}\left\{\eta_0 \pm \frac{y}{3\pi\alpha^3}\right\} + \ldots = \epsilon^{\frac{3}{2}}\left\{\eta_0 - \frac{2^3}{3\pi^4 y^5}(1 \pm [1 + \tfrac{1}{2}\pi^2 y^2 x]^{\frac{1}{2}})^3\right\} + \ldots. \quad (4.31)$$

These solutions are valid for $x > 0$ if $v_0(r_0) < 0$, and for $x > x_e$ if $v_0(r_0) > 0$. As $\alpha \to -\infty$,

$$x_e = (\pi\alpha)^{-1} + \ldots, \quad y_e = -(-2\alpha/\pi)^{\frac{1}{2}} + \ldots, \quad (4.32)$$

and therefore $x_e = -2/(\pi^2 y^2) < -3/(2\pi^2 y^2)$. As expected, the envelope extends into the undisturbed region beyond the shock. Comparing (4.30) and (4.31) with (4.14), we see that the matching condition $(1-I)/(1-I_L) = (1-I_L)/(1-I)$ requires that $\eta_0 = 0$.

Proceeding as in the above paragraph, it can be shown that (4.25) with $\eta_0 = 0$ satisfies first-order matching conditions with all the contiguous expansions of §§2 and 3; the remaining constants in the inner solution are not determined by formal matching. The applicability of the matching principle can be shown by simple examples to depend upon an appropriate choice of outer and inner variables. As noted at the beginning of §4.2, the first-order inner solution ahead of the shock or characteristic is precisely the $(1-I)/(1-I_M)$ expansion. However, if it is desired to carry out formal first-order matching between the inner and main front expansions, it is found necessary to introduce new inner variables, e.g. $X_1 = [1 - r\cos(\theta_0 - \theta)]/\epsilon$ and $Y_1 = (\theta_0 - \theta)/\epsilon^{\frac{1}{2}}$ to replace X and Y. Then the inner expansion in the variables X_1 and Y_1 matches formally to first order with the I_M expansion.

Thus, (4.25) becomes a two-parameter family of exact solutions to the field equations (4.13) appropriate to the inner region for both cases $v_0(r_0) \gtrless 0$, which satisfies all matching conditions that can be imposed at this stage of the expansion process. It remains to satisfy the two shock conditions (4.12), but with only the two arbitrary constants k_3 and y_0 in the solution, it is to be expected that the two conditions cannot be satisfied simultaneously. For the case $v_0(r_0) < 0$, it does not appear possible to construct shocks satisfying (4.12) in the region near $y = 0$. On the other hand, for $v_0(r_0) > 0$, k_3 and y_0 can be chosen so that both shock conditions are approximately satisfied. The construction of the shock and the corresponding field behind the shock for $v_0(r_0) > 0$ represents the major result of this analysis, and this case is considered in detail in §5.

5. Approximate construction of the shock when $v_0(r_0) > 0$

In the case of compressive flow behind the main front, there is a shock near that front, as well as a shock near the diffracted front to the left of the singular point. We assume that these are segments of a single shock wave which continues through the inner region and separates undisturbed flow ahead of the shock from disturbed flow behind it. Under this assumption, the two shock conditions (4.12) become

$$dx/dy = -\eta/\xi \quad \text{and} \quad dx/dy = -\{-(2x+\xi)\}^{\frac{1}{2}}, \quad (5.1a, b)$$

where ξ and η are evaluated directly behind the shock.

Further, in this case there is a characteristic of discontinuity near the diffracted front to the right of the singular point, whose equation is given by

$$(1-r) = \epsilon E(\theta) + o(\epsilon)$$

(see (3.11)). Forming the one-term inner expansion of the above equations, we obtain $X = E(\theta_0)$, or $x = -1$ as the leading term of the inner representation of this characteristic. The solution (4.26) is defined for $x > -1$, $y > y_0$. However, if $x = -1$ is to represent a characteristic of discontinuity, the inner solution consisting of (4.26) and (4.21) in their respective regions of definition must be continuous across this characteristic. To satisfy this condition for η as well as for ξ, y_0 must be set equal to zero.

We consider the conditions (5.1) from the following point of view: if a field (ξ, η) satisfying (4.13) and appropriate matching conditions is assumed to exist behind the shock, then each of conditions (5.1) together with an appropriate initial condition furnishes an equation for determination of the shock. If the assumed field is the correct solution to the first-order inner problem, then the two curves computed from (5.1 *a*, *b*) must coincide, and this coincidence can, in effect, be taken as a criterion for the correctness of the field and corresponding shock wave as a solution to the first-order inner problem.

Applying this criterion of coincidence in the region ahead of the characteristic of discontinuity for large positive y, where the only available candidate for the field behind the shock is $\xi = 1, \eta = y$, we immediately determine the shock as

$$x + \tfrac{1}{2}y^2 = -\tfrac{1}{2}. \tag{5.2}$$

(This is exactly the leading term of the inner expansion of the shock associated with the I_M expansion.) With both shock conditions satisfied on the shock (5.2), we are led to accept this shock and the field $\xi = 1, \eta = y$ as the leading term of the inner expansion of the exact solution in this region.

The shock (5.2) intersects the characteristic of discontinuity $x = -1$, when $y = 1$. For $y < 1$, the available candidates for the field are the family of particular solutions of the inner equations (4.25) with $y_0 = 0$,

$$x = \tfrac{1}{2}\eta^2\phi'(\xi) - g(\xi; k_3), \quad y = \eta\phi(\xi), \tag{5.3}$$

all of which satisfy matching conditions with the contiguous expansions of §3 and with the outer expansion. An initial condition for the integration of (5.1) is provided by the fact that $x = -1$, $y = 1$ is a point on the shock. Since (5.3) gives x and y explicitly in terms of ξ and η, while ξ and η can only be represented implicitly in terms of x and y, it is convenient to study the image of the shock in the hodograph plane. From (5.1), the following conditions must be satisfied on the hodograph image of the shock:

$$d\eta/d\xi = -(\xi x_\xi + \eta y_\xi)(\xi x_\eta + \eta y_\eta)^{-1}, \tag{5.4a}$$

$$d\eta/d\xi = -\{x_\xi + y_\xi(-(2x+\xi))^{\frac{1}{2}}\}\{x_\eta + y_\eta(-(2x+\xi))^{\frac{1}{2}}\}^{-1}. \tag{5.4b}$$

The initial condition in the hodograph plane follows from the fact that the field immediately behind the shock at the point $x = -1$, $y = 1$ is $\xi = 1$, $\eta = 1$.

For any solution of (4.13) $x_\eta = y_\xi$, which implies that (5.4*a*) is an exact equation whose general solution may immediately be written in the form

$$\xi x + \eta y - \left\{ \int_1^\xi x(\bar{\xi}, \eta)\, d\bar{\xi} + \int_1^\eta y(1, \bar{\eta})\, d\bar{\eta} \right\} = \text{constant}. \tag{5.5}$$

Inserting (5.3), carrying out the integrations, and imposing the initial condition $\eta(1) = 1$, we obtain

$$\tfrac{1}{2}\eta^2\{\xi\phi(\xi)\}' - (2\pi^2)^{-1}\{\pi^2\xi^2 + \pi\xi \sin(2\pi\xi) + \tfrac{1}{2}(\cos(2\pi\xi) - 1\} + (k_3/4\pi)\{2\pi(\xi\cos(2\pi\xi) - 1) - \sin(2\pi\xi)\} = 0. \tag{5.6}$$

As $\xi \to 0$ on the curves (5.6), we have

$$\eta^2 = \tfrac{1}{6}k_3\pi^2\xi^3 + \ldots \quad \text{if} \quad k_3 \neq 0, \quad \text{and} \quad \eta^2 = \tfrac{1}{3}\pi^2\xi^5 + \ldots \quad \text{if} \quad k_3 = 0.$$

Therefore, since $x = (9\eta^2)/(2\pi^2\xi^4) - 2\xi + \ldots$ and $y = -(3\eta)/(\pi^2\xi^3) + \ldots$ as $\xi \to 0$, on the curves (5.6)

$$x = \tfrac{1}{4}3k_3\xi^{-1} + \ldots, \quad y = -(3k_3^{\frac{1}{2}}\pi^{-\frac{3}{2}}6^{-\frac{1}{2}})\,\xi^{-\frac{3}{2}} + \ldots \quad \text{if} \quad k_3 \neq 0,$$
$$x = -\tfrac{1}{2}\xi + \ldots, \quad y = -(3^{\frac{1}{2}}\pi^{-1}), \quad \xi^{-\frac{1}{2}} + \ldots \quad \text{if} \quad k_3 = 0.$$

From (4.14) we see that the shock should have the asymptotic behaviour $x = -3/(2\pi^2y^2)$ as $y \to -\infty$. Therefore, (5.6) can represent the shock in the inner region only if $k_3 = 0$, so that (5.6) has the correct asymptotic behaviour to match the shock associated with the I_L expansion. With k_3 set equal to zero, we find for the field

$$x = \tfrac{1}{2}\eta^2\phi'(\xi) - g(\xi; 0), \quad y = \eta\phi(\xi), \tag{5.7}$$

and, for the shock satisfying (5.4*a*),

$$\tfrac{1}{2}\eta^2\{\xi\phi(\xi)\}' - (2\pi^2)^{-1}\{\pi^2\xi^2 + \pi\xi\sin(2\pi\xi) + \tfrac{1}{2}(\cos(2\pi\xi) - 1)\} = 0. \tag{5.8}$$

It remains to consider shock condition (5.4*b*) subject to the initial condition $\eta(1) = 1$. We note that, from (5.8), $d^2\eta/d\xi^2 = -2 - \pi^2$ at the point $\xi = 1$, $\eta = 1$, while (5.4*b*) and (5.7) imply that the shock computed using (5.7) and the second shock condition will have $d^2\eta/d\xi^2 = -3 - \pi^2$ at $\xi = 1, \eta = 1$. Therefore, the shock computed using (5.4*b*) and the field (5.7) will not coincide exactly with the shock (5.8). This discrepancy in the solution is attributable to the fact that a general solution to the system of inner equations (4.11) is not available, and the family of particular solutions (5.3) is too restricted to yield an exact solution to the first-order inner problem.

However, the following considerations show that the result (5.7) and (5.8) is a remarkably good approximation to the first-order inner solution. Equation (5.4*b*) with the field (5.7) may be integrated numerically to any desired degree of accuracy and the results of such a numerical integration are shown in figure 4, together with a graph of (5.8).† Although the two curves do not coincide every-

† Here we point out that, if the field (5.3) is used in conjunction with the shock condition (5.4*b*), both numerical computations and an examination of the direction field associated with this differential equation near the origin indicate that as $\xi \to 0$ $\eta = \pi^{3-\frac{1}{2}}\xi^{\frac{5}{2}} + \ldots$ for all values of k_3 in some neighbourhood of $k_3 = 0$. This is the correct behaviour to match the shock associated with the I_L expansion, since this $\eta(\xi)$ is precisely the hodograph image of the shock obtained from the $(1-I)/(1-I_L)$ expansion.

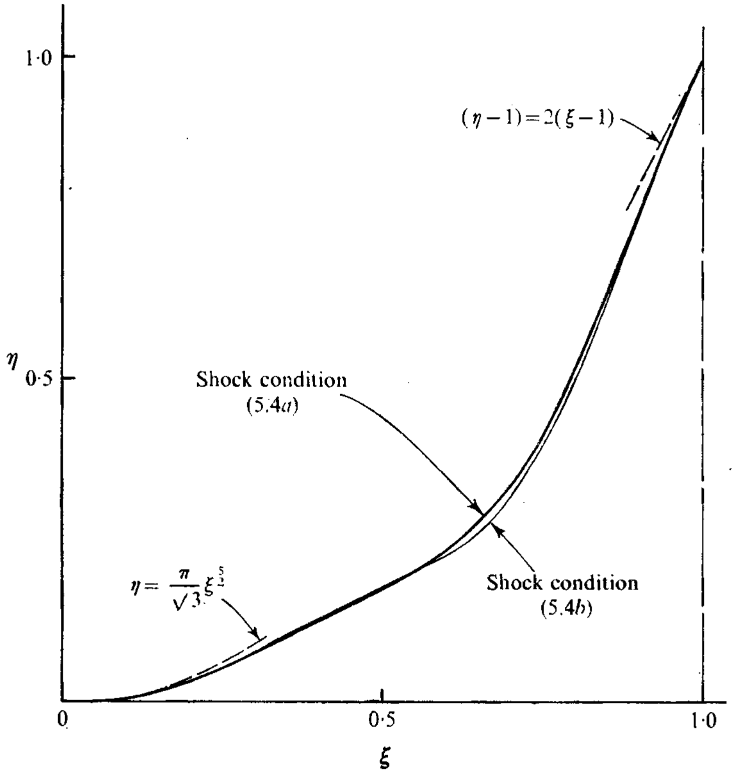

FIGURE 4. Hodograph image of shock associated with the field:
$x = \frac{1}{2}\eta^2\phi'(\xi) - g(\xi; 0), y = \eta\phi(\xi)$.

where, they are asymptotic to each other and to the correct shock wave near the points (0, 0) and (1, 1) in the hodograph plane. Further, they differ from one another by less than about 6 % over $0 \leqslant \xi \leqslant 1$, with the maximum difference occurring at two points near $\xi = 0{\cdot}4$ and $\xi = 0{\cdot}7$. To estimate the error in this approximation, a more accurate analytical or numerical solution to the inner problem is required. It appears that such an improvement would require considerable effort to obtain a correction which is expected to be uniformly small.

6. Results and discussion

The results of our analysis for the case $v_0(r_0) > 0$ (compression behind the main front) are summarized in figure 5, where the perturbation pressure field according to our approximate first-order inner solution is shown superimposed on the first-order outer (linearized theory) perturbation pressure field. The shock resulting from each of the conditions (5.1 *a, b*) is also shown. The results exhibited in this figure are quite general, and apply to conical bodies of arbitrary shape and orientation, within the restrictions imposed in (2.6), (2.7). These results have been specialized for a specific physical example of a flat plate at incidence, and are presented infigure 6. For this case, $G(\hat{x}) = \beta$, $r_1 = 0$, $r_0 = +\infty$; the Mach number is chosen to be $M = 2$ and $\tan^{-1}\epsilon$, the angle of incidence, is taken as 2°.

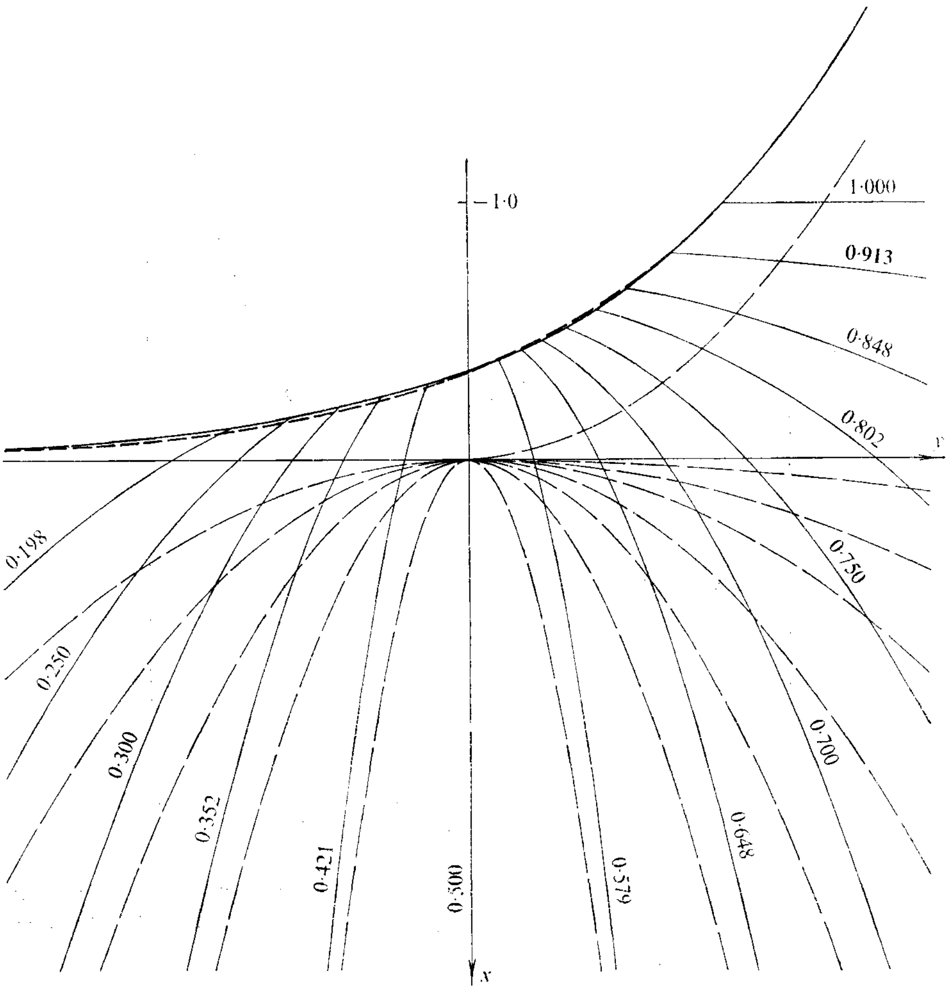

FIGURE 5. Perturbation pressure field and shock when $v_0(r_0) > 0$. ——, $p/(\epsilon C_0 M^2)$ = constant (first-order inner); – –, $p/(\epsilon C_0 M^2)$ = constant (first-order outer linearized theory); ——, shock (5.1*a*); – –, shock (5.1*b*).

$$x = \frac{1-r}{\epsilon\beta C_0 K}, \quad y = \frac{\theta_0-\theta}{(\epsilon\beta C_0 K)^{\frac{1}{2}}}, \quad p = \frac{\bar{p}-p_0}{c_0^2\rho_0}.$$

The inner solution provides a smooth transition between the flow near the main front, where the perturbation quantities and shock strength are $O(\epsilon)$, and the flow near the diffracted front, where these quantities are $O(\epsilon^2)$. The shocks computed from both shock conditions are shown in figure 5, and these curves are seen to be almost coincident everywhere, a noticeable departure occurring only for a small range of negative y. The differences are insignificant in the physical variables on the scale of figure 6. Lines of constant perturbation pressure according to linearized theory are shown in both figures. The directional singularity of the linearized solution reflects the behaviour of the inner solution as one moves away from the singular point, but sufficiently close to that point the linearized

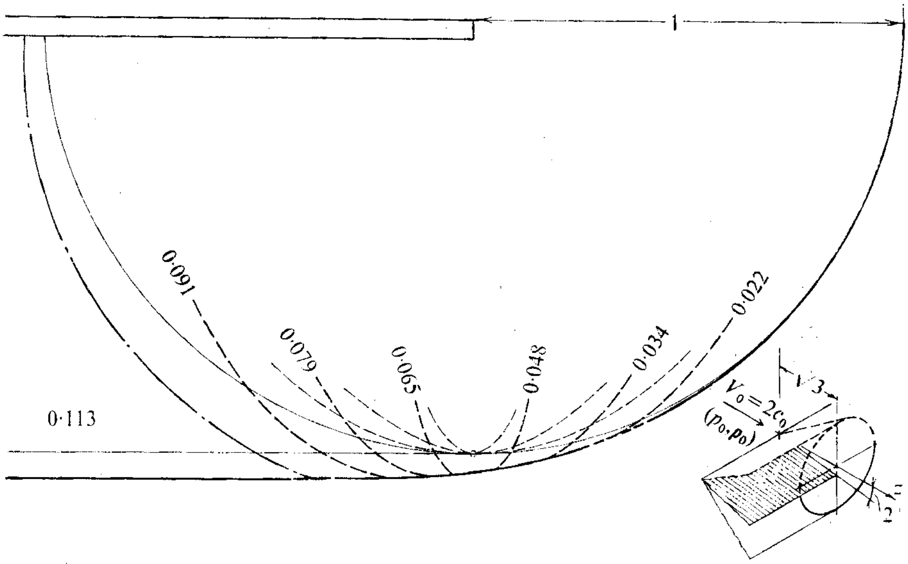

FIGURE 6. Pressure field and shock below a flat, rectangular plate at incidence. $\gamma = 1{\cdot}4$, $M = 2$. ——, shock; — — —, characteristic of discontinuity; – –, $(\bar{p}-p_0)/p_0$ = constant; ——, – –, linear theory.

theory is completely inaccurate. Although the range of perturbation pressures predicted by the inner solution is the same as that of the linearized solution ($0 < p/\epsilon C_0 M^2 < 1$), the distribution of pressure differs radically in the two solutions. Figure 6 shows that, while the directional singularity of the linearized solution predicts an instantaneous decay in the perturbation pressure behind the wave front from $p/\epsilon C_0 M^2 = 1$ to $p/\epsilon C_0 M^2 = 0$ as one crosses $\theta = \theta_0$, in the inner solution this decay occurs much more gradually, requiring an angular span of about 40° in this case. Finally, the inner solution has shock discontinuities for $\theta > \theta_0$ whereas the linearized solution predicts a continuous field across the wave front, and the inner solution predicts an expansion at a finite rate behind the characteristic of discontinuity, whereas the linearized solution shows an expansion at an infinite rate. The characteristic of discontinuity has been moved outward from its linearized position, in accordance with the fact that the sound speed is increased over its free-stream value in the compressed region behind the main front.

As was mentioned in § 1, the nature of the flow in the vicinity of the singular point was the subject of a number of previous investigations. Legras (1953) tried to generalize the method of Lighthill (1949*b*), to determine the flow near the singular points associated with a rectangular plate at incidence, such as that shown in figure 6. However, his analysis seems incorrect: he found a plane shock above the plate, where in fact a continuous Prandtl–Meyer expansion fan must exist. Further, he found a discontinuity in the perturbation pressure immediately behind the shock at the point where the plane shock below the plate intersects the characteristic of discontinuity. Such a discontinuity implies the existence of another shock branching out from this point. Our solution shows that it is not necessary to assume such a shock configuration: there exists only a single, smooth shock, and the pressure varies continuously behind it.

Davis (1969) attempted to extend the methods introduced by Whitham (1952, 1956), to include the behaviour of the flow near the singular points. While the general problem treated in Davis (1969) involves a body of finite dimensions, there exist regions where the flow is conical (near the wing-tips), and Davis (1969) provides formulae for the axial perturbation velocity field w and for the shock in these regions. However, this approximation for w is not the first term of a uniformly valid (composite, generalized) asymptotic expansion in the sense of Van Dyke (1964), because it can be shown that the one-term inner expansion of the approximation yields a $W^{(1)}(X, Y)$ which is incompatible with the inner equations. On the other hand, a comparison of the one-term inner expansion of the approximation for w given in Davis (1969) with our approximate solution for the same body shape (a wedge of rectangular plane form with the plane $\theta = 0$ bisecting the wedge) shows that these two functions do not differ greatly. This would partially explain why there is good agreement between the analytical results of Davis (1969) and the experimental results of Davis (1971).

When the work presented here was substantially complete, the book by Bagdoyev (1969) appeared, which dealt primarily with unsteady shock wave propagation in compressible liquids. Bagdoyev found solutions essentially equivalent to (5.3) for the unsteady diffraction problem analogous to the problem treated here. However, he applied only the second of the shock conditions (5.1), apparently ignoring the first. But, as noted in §5, this makes the solution to the inner problem indeterminate: a shock can be constructed using (5.1*b*) satisfying the initial condition $x(1) = -1$, and having the correct asymptotic behaviour as $y \to -\infty$ for a range of values of k_3 around $k_3 = 0$. The author does not mention this difficulty and appears to make the arbitrary choice $k_3 = \frac{1}{2}$, but, as shown in §5, this field would produce an unacceptable shock when used in conjunction with (5.1*a*). In fact, as Hayes (1971) remarked, there are two independent, equally important, shock conditions which the correct inner solution (ξ, η) must satisfy.

Seebass (1971) discussed a related problem involving nonlinear propagation in the vicinity of a caustic of linearized theory. By means of a hodograph-related transformation, he succeeded in linearizing the inner equations of motion and obtaining their solution subject to appropriate asymptotic boundary conditions. As in the problem considered here, Seebass' solution allows exact satisfaction of only one of the shock conditions. However, in contrast to the results given here, the second independent shock relation can be approximately satisfied by Seebass' solution to the caustic problem only when the shock strength is infinitesimal.

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